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Continuity Properties of Chebyshev Centers*

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1. INTRODUCTION AND NOTATIONS

Let X be a Banach space, F a bounded closed subset of X, V a closed subset of X. A point $x \in V$ is said to be a relative Chebyshev center of F with respect to V if x is the center of the smalest closed ball with center in V containing F, i.e., if

$$x \in \{z \in V; \sup_{y \in F} \| z - y \| = \mathrm{rad}_V(F)\}, \quad \text{where } \mathrm{rad}_V(F) = \inf_{w \in V} \sup_{y \in F} \| w - y \|.$$

The number $\mathrm{rad}_V(F)$ is called the relative Chebyshev radius of F with respect to V. We denote the set of all such Chebyshev centers by $\mathrm{cent}_V(F)$. The question of the existence, unicity and stability of Chebyshev centers has been recently studied by several authors (cf., e.g., [8, 13, 14, 21–23]).

In this paper we study the continuity properties of cent_V. This is clearly a set-valued function from 2^X into 2^V (we assume 2^X to be equipped with the Hausdorff metric d). We show here that cent_V is an upper Hausdorff semicontinuous function if X is an arbitrary Banach space and V is a finite-dimensional closed convex subset of X, and if $X = l_1$ and V is a w^*-closed convex subset of X. We show further that cent_V is Hausdorff continuous on the subclass $\mathscr{K}(X)$ of 2^X of all compact subsets of X if X is a dual locally uniformly convex (l.u.c.) Banach space and V is a w^*-closed convex subset of X, and if X is a Lindenstrauss space and V is an M-ideal in X.

Let S be a compact Hausdorff space, $C(S, X)$ the space of all continuous functions on S with values in a Banach space X equipped with the norm of the uniform convergence. A subspace V of $C(S, X)$ is said to be a Stone–Weierstrass (SW-)subspace of $C(S, X)$ if there is a compact Hausdorff space T and a continuous surjection φ from S onto T such that

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$V = \{f \in C(S, X); f = g \circ \varphi$ for some $g \in C(T, X)\}$. Mazur (unpublished, cf., e.g., [19]) proved that such subspaces are proximinal if $X = \mathbb{R}$ (a subspace G of a normed linear space X is called proximinal if every $x \in X$ possesses a best approximation in G). Pelczynski [17] asked whether for a given Banach space X every SW-subspace of $C(S, X)$ is proximinal. Olech [16] and Blatter [3] showed that this conjecture is true if X is a uniformly convex Banach space and a Lindenstrauss space, respectively (a Lindenstrauss space is a space whose dual is $L_1(\mu)$ for some measure μ). Lau [10] showed that for X uniformly convex this result remains true even if the assumption of the compactness of S and T is dropped. Here we give a contribution to this problem. By the application of our previous results we show that every SW-subspace with φ open is proximinal if X is a dual l.u.c. Banach space. Further, we give an example of a Banach space X for which the answer to Pelczynski's question is negative.

We employ the following notations. $\mathbb{R}$ and $\mathbb{N}$ will denote the set of all real numbers and the set of all positive integers, respectively. Let X be a Banach space, $x \in X$, $r > 0$. $B(x, r)$ will denote the closed ball in X with center x and radius r. A set-valued function f from a topological space S into 2^X is called upper Hausdorff semicontinuous (u.H.s.c.) respectively lower Hausdorff semicontinous (l.H.s.c.) if for every $s_0 \in S$ and every $\epsilon > 0$ there is a neighborhood U of s_0 such that for every $s \in U$ we have $\sup_{x \in f(s)} \operatorname{dist}(x, f(s_0)) \leqslant \epsilon$ respectively $\sup_{x \in f(s_0)} \operatorname{dist}(x, f(s)) \leqslant \epsilon$. The function f is Hausdorff continuous (H.c) if f is both u.H.s.c. and l.H.s.c. The function f is u.s.c. respectively l.s.c. if it is upper semicontinuous respectively lower semicontinuous in the usual sense (cf. [18, 20]). A Banach space X is said to be locally uniformly convex (l.u.c.) if for every $x \in X$ with $\| x \| = 1$ and every sequence $\{y_n\} \subset X$ with $\lim \| y_n \| \leqslant 1$, $\lim \|(x + y_n)/2\| \geqslant 1$ implies $\lim \|x - y_n\| = 0$. X is said to be uniformly convex in every direction (u.c.e.d.) (cf., e.g., [6, 8]) if for every $\epsilon > 0$ and every $z \in X$ there is a $\delta > 0$ such that $\| x_1 \| = \| x_2 \| = 1$, $x_1 - x_2 = \lambda z$ for some $\lambda \in \mathbb{R}$ and $\|(x_1 + x_2)/2\| \geqslant 1 - \delta$ implies $|\lambda| \leqslant \epsilon$. All Banach spaces in this paper are real.

2. Semicontinuity of cent_V

In this section we study the upper and lower Hausdorff semicontinuity of cent_V. To avoid ad hoc proofs and to simplify the exposition the following definition appears useful.

Definition. Let X be a Banach space, $\mathfrak{A}$ a class of closed bounded subsets of X, V a closed subset of X. The pair $(V, \mathfrak{A})$ is said to have the property P_1 if for every $F \in \mathfrak{A}$ and every $\epsilon > 0$ there is a $\delta > 0$ such that for every $x \in \bigcap_{y \in F} B(y, \mathrm{rad}_V(F) + \delta) \cap V$ we have $\operatorname{dist}(x, \bigcap_{y \in F} B(y, \mathrm{rad}_V(F)) \cap V) < \epsilon$.

The pair $(V, \mathfrak{A})$ is said to have the property P_2 if it has the property P_1 such that $\delta > 0$ can be chosen independently on $F \in \mathfrak{A}$. We use the convention $\operatorname{dist}(x, \varnothing) = +\infty$ here.

Now, we give some examples.

PROPOSITION 1. *Let X be an arbitrary Banach space, V a finite-dimensional closed convex subset of X, $\mathfrak{A}$ the class of all bounded, closed, non-empty subsets of X. Then the pair $(V, \mathfrak{A})$ has the property P_1.*

The proof is easy and is left to the reader. To prove Proposition 2 we need the following lemma. Its proof may be found in [12].

LEMMA. *Let $\{x_n\} \subset l_1$ be a sequence weakly* converging to 0. Let $y \in l_1$. Then for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $n \geqslant n_0$ implies $| \| x_n - y \| - \| x_n \| - \| y \| | < \epsilon$.*

PROPOSITION 2. *Let $X = l_1$. Let V be a w^*-closed convex subset of X, $\mathfrak{A}$ the class of all bounded closed non-empty subsets of X. Then the pair $(V, \mathfrak{A})$ has the property P_1.*

Proof. Assume the contrary. Then there is an $\epsilon_0 > 0$ and a set $F \in \mathfrak{A}$ such that for every $n \in \mathbb{N}$ there exists an element $z_n \in V$ such that $z_n \in \bigcap_{y \in F} B(y, \operatorname{rad}_V(F) + 1/n)$ and $\operatorname{dist}(z_n, \bigcap_{y \in F} B(y, \operatorname{rad}_V(F)) \cap V) \geqslant \epsilon_0$. Without loss of generality we may assume $w^* - \lim z_n = 0$. It is impossible that $\lim \| z_n \| = 0$, so $\eta_0 = \limsup \| z_n \| > 0$. For every $y \in F$ we obviously have $\limsup \| y - z_n \| \leqslant \operatorname{rad}_V(F)$. Let $\epsilon > 0$ be given. Then for every $n \in \mathbb{N}$ sufficiently big we have $\| z_n - y \| < \operatorname{rad}_V(F) + \epsilon/3$ and, by the previous lemma, $| \| z_n - y \| - \| z_n \| - \| y \| | < \epsilon/3$. On the other hand there is a subsequence $\{z_{n_k}\}$ with $\| z_{n_k} \| \geqslant \eta_0 - (\epsilon/3)$ for each $k \in \mathbb{N}$. Thus for every $y \in F$ and suitable $k \in \mathbb{N}$ we have

$$\begin{aligned} \| y \| &\leqslant \| z_{n_k} - y \| - \| z_{n_k} \| + \epsilon/3 \\ &\leqslant \operatorname{rad}_V(F) + 2\epsilon/3 - \| z_{n_k} \| \leqslant \operatorname{rad}_V(F) - \eta_0 + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ has been arbitrary we have $\| y \| \leqslant \operatorname{rad}_V(F) - \eta_0$ for every $y \in F$. This, however, implies $B(0, \eta_0) \subset B(y, \operatorname{rad}_V(F))$ for every $y \in F$. Thus $B(0, \eta_0) \cap V \subset \bigcap_{y \in F} B(y, \operatorname{rad}_V(F)) \cap V$. But $\lim \operatorname{dist}(z_n, B(0, \eta_0) \cap V) = 0$. A contradiction.

A closed subspace V of a Banach space X is called an M-ideal if there exists a projection P on the dual X^* of X onto $V^\perp$, the annihilator of V, such that for every $u \in X^*$ we have $\| u \| = \| Pu \| + \| u - Pu \|$. The concept of an M-ideal has been introduced and studied in [1] (cf. also [2, 7, 9]). It has been shown in [13] that $\operatorname{cent}_V(F) \neq \varnothing$ for every compact subset F of a Lindenstrauss space X and every M-ideal V.

PROPOSITION 3. *Let X be a Lindenstrauss space, V an M-ideal in X and $\mathfrak{A}$ the class of all compact non-empty subsets of X. Then the pair $(V, \mathfrak{A})$ has the property P_2.*

Proof. Put $\delta = \epsilon$. Let $F \in \mathfrak{A}$, $x \in \bigcap_{y \in F} B(y, \operatorname{rad}_V(F) + \delta) \cap V$. Then obviously $B(x, \delta) \cap B(y, \operatorname{rad}_V(F)) \neq \varnothing$ for every $y \in F$. Since also $\bigcap_{y \in F} B(y, \operatorname{rad}_V(F)) \cap V = \operatorname{cent}_V(F) \neq \varnothing$, the balls $B(y, \operatorname{rad}_V(F))$, $y \in F$, $B(x, \delta)$ intersect pairwise. By a well-known theorem of Lindenstrauss [11] $B(x, \delta) \cap \bigcap_{y \in F} B(y, \operatorname{rad}_V(F)) \neq \varnothing$. Further, each of the above balls intersects V. The rest of the proof follows from the next lemma [13].

LEMMA. *Let X, V and $\mathfrak{A}$ be as in Proposition 3. Let $K \in \mathfrak{A}$, $r > 0$. Assume that $B(x, r) \cap V \neq \varnothing$ for every $x \in K$ and that $\bigcap_{x \in K} B(x, r) \neq \varnothing$. Then $\bigcap_{x \in K} B(x, r) \cap V \neq \varnothing$.*

Garkavi [8] showed that a Banach space X is u.c.e.d. if and only if for every bounded set $F \subset X$ $\operatorname{cent}_X(F)$ consists of at most one element. The same argument with obvious modifications shows that if X is strictly convex then $\operatorname{cent}_V(F)$ consists of at most one element for every compact set $K \subset X$ and every convex closed set $V \subset X$.

PROPOSITION 4. *Let X be a l.u.c. space. Let V be a closed convex subset of X, $\mathfrak{A}$ the class of all compact non-empty subsets of X. Let $\operatorname{cent}_V(F) \neq \varnothing$ for every $F \in \mathfrak{A}$. Then the pair $(V, \mathfrak{A})$ has the property P_1.*

Proof. Assume the contrary. Then there is a compact set $F \subset X$ and an $\epsilon_0 > 0$ such that for every $n \in \mathbb{N}$ there exists an $x_n \in V$ such that $x_n \in \bigcap_{y \in F} B(y, \operatorname{rad}_V(F) + 1/n)$ and $\| x_n - x_0 \| \geqslant \epsilon_0$, where $x_0 = \operatorname{cent}_V(F) = \bigcap_{y \in F} B(y, \operatorname{rad}_V(F)) \cap V$. Put $w_n = (x_n + x_0)/2$, $n \in \mathbb{N}$. Since w_n cannot be in $\operatorname{cent}_V(F)$ for every $n \in \mathbb{N}$ there exists a $y_n \in F$ with $\| w_n - y_n \| > \operatorname{rad}_V(F)$. Without loss of generality we may assume that $\lim y_n = y_0$ for some $y_0 \in F$. For every $n \in \mathbb{N}$ denote $\epsilon_n = \| y_n - y_0 \|$. Then we have

$$\| y_0 - w_n \| \geqslant \| y_n - w_n \| - \| y_n - y_0 \| > \operatorname{rad}_V(F) - \epsilon_n$$

and

$$\| y_0 - x_n \| \leqslant \| y_0 - y_n \| + \| y_n - x_n \| \leqslant \operatorname{rad}_V(F) + 1/n + \epsilon_n .$$

It follows that for suitable subsequences we have $\| u_0 \| \leqslant \operatorname{rad}_V(F)$, $\lim \|v_k\| \leqslant \operatorname{rad}_V(F)$ and $\lim \|(u_0 + v_k)/2\| \geqslant \operatorname{rad}_V(F)$, where $u_0 = y_0 - x_0$, $v_n = y_0 - x_n$, which together with $\| u_0 - v_n \| = \| x_n - x_0 \| \geqslant \epsilon_0$, $n \in \mathbb{N}$, contradicts the assumption that X is locally uniformly convex.

Remark. If X is a uniformly convex Banach space then in the previous proposition $\mathfrak{A}$ can be taken to be the class of all closed bounded non-empty subsets of X. The proof is similar to that of Proposition 4 and is left to the reader.

The assumptions of Proposition 4 are fulfilled, e.g., for all dual l.u.c. Banach spaces and all w^*-closed convex subsets V of X. This is an immediate consequence of Alaoglu's theorem.

Now, we establish the connection between the properties P_1 and P_2 and the Hausdosff semicontinuity of cent_V.

THEOREM 5. *Let X be a Banach space, V a closed subset of X and $\mathfrak{A}$ a class of bounded closed non-empty subsets of X. If the pair $(V, \mathfrak{A})$ has the property P_1 then the function* cent_V *is u.H.s.c. on* $\mathfrak{A}$.

Proof. Let $F \in \mathfrak{A}$ and $\epsilon > 0$ be given. Take the corresponding $\delta > 0$. It is easy to show that $\text{cent}_V(G) \subset \bigcap_{y\in F} B(y, \text{rad}_V(F) + \delta) \cap V$ for every $G \in \mathfrak{A}$ with $d(F, G) < \delta/2$. Indeed, $x \in G$ implies $\text{dist}(x, F) < \delta/2$. Hence $\text{rad}_V(G) < \text{rad}_V(F) + \delta/2$. Similarly $\text{rad}_V(F) < \text{rad}_V(G) + \delta/2$. Let $y \in F$. Then there is an $x_y \in G$ with $\| y - x_y \| < \delta/2$. For every such pair we have $B(x_y, \text{rad}_V(G)) \subset B(y, \text{rad}_V(G) + \delta/2) \subset B(y, \text{rad}_V(F) + \delta)$. This implies $\text{cent}_V(G) = \bigcap_{x\in G} B(x, \text{rad}_V(G)) \cap V \subset \bigcap_{y\in F} B(x_y, \text{rad}_V(G)) \cap V \subset \bigcap_{y\in F} B(y, \text{rad}_V(F) + \delta) \cap V$. Since the pair $(V, \mathfrak{A})$ has the property P_1 we have $\text{dist}(x, \text{cent}_V(F)) < \epsilon$ for every $x \in \text{cent}_V(G)$.

THEOREM 6. *If the pair $(V, \mathfrak{A})$ has the property P_2 then* cent_V *is H.c. on* $\mathfrak{A}$.

Proof. Since P_2 implies P_1 we have only to show that cent_V is l.H.s.c. on $\mathfrak{A}$. Let $F \in \mathfrak{A}$ and $\epsilon > 0$ be given. Take the corresponding $\delta > 0$. It is clear from the proof of Theorem 5 that $\text{cent}_V(F) \subset \bigcap_{y\in G} B(y, \text{rad}_V(G) + \delta) \cap V$ for every $G \in \mathfrak{A}$ with $d(F, G) < \delta/2$. Hence $\text{dist}(x, \text{cent}_V(G)) < \epsilon$ for every such G and every $x \in \text{cent}_V(F)$.

Remark. The property l.H.s.c. is obviously stronger than the usual lower semicontinuity. Thus, by Michael's selection theorem [15], cent_V admits a continuous selection on $\mathfrak{A}$ if the pair $(V, \mathfrak{A})$ has the property P_2.

COROLLARY 7. *Let X be a Banach space, V a closed subset of X, $\mathfrak{A}$ a class of bounded closed non-empty subsets of X. Then* cent_V *is u.H.s.c. on* $\mathfrak{A}$ *if one of the following conditions is fulfilled.*

(i) *V is convex and finite-dimensional,*

(ii) *$X = l_1$ and V is convex and w^*-closed,*

(iii) *X is uniformly convex and V is convex,*

(iv) *X is a dual l.u.c. Banach space, V is w^*-closed convex and the sets $\mathfrak{A}$ are all compact,*

(v) *X is a Lindenstrauss space, V is an M-ideal in X and the sets in* $\mathfrak{A}$ *are all compact. In this case* cent_V *is even H.c.*

Cheney and Wulbert [4] have given an example of a Chebyshev subspace V of l_1 for which the metric projection P_V is discontinuous. The subspace V of their example is obviously not w^*-closed. Since the metric projection coincides with cent_V on the class of all singletons $\{f\} \subset l_1$ the next result follows from Corollary 7(ii).

COROLLARY 8. *Let V be a w^*-closed Chebyshev subspace of l_1. Then the metric projection P_V is continuous.*

Remark. Since each of the assumptions (iii) and (iv) of Corollary 7 implies that $\mathrm{cent}_V(F)$ consists of exactly one element for every bounded set $F \subset X$, it follows from Corollary 7 that in both these cases cent_V is continuous on $\mathfrak{A}$.

3. PROXIMINALITY OF STONE–WEIERSTRASS SUBSPACES

Let S and T be compact Hausdorff spaces, $\varphi: S \to T$ a continuous surjection, $V = \{f \in C(S, X); f = g \circ \varphi$ for some $g \in C(T, X)\}$. For every $f \in C(S, X)$ denote $\Phi_f(t) = \{f(s); s \in \varphi^{-1}(t)\}$, $t \in T$. The following theorem gives a sufficient condition for the existence of a best approximation in V.

THEOREM 9. *Let V be a SW-subspace of $C(S, X)$ such that the corresponding function φ is open. Let $f \in C(S, X)$. If* cent_X *admits a continuous selection on the class $\mathscr{K}(X)$ of all non-empty compact subsets of X then there exists a best approximation of f in V.*

Proof. It is easy to see that $\mathrm{dist}(f, V) \geqslant \sup_{t\in T} \mathrm{rad}_X(\Phi_f(t))$. Olech [16] showed that Φ_f is a u.s.c. function which implies that Φ_f is u.H.s.c. It is easy to show that Φ_f is l.s.c. Indeed, let $t_0 \in T$, $x = f(s) \in \Phi_f(t_0)$ and $\epsilon > 0$ be given. Then there is an open neighborhood U of s with $f(s') \subset B(x, \epsilon)$ for every $s' \in U$. It follows that $\Phi_f(t) \cap B(x, \epsilon) \neq \varnothing$ for every point $t \in U' = \varphi(U)$. Since $\Phi_f(t)$ is compact for every $t \in T$ Φ_f is l.H.s.c.

Now, let $h: \mathscr{K}(X) \to X$ be a continuous selection of cent_X. Define $g = h \circ \Phi_f \circ \varphi$. The function g is obviously continuous and we have

$$\|f - g\| = \sup_{s\in S} \|f(s) - g(s)\| = \sup_{t\in T} \sup_{s\in\varphi^{-1}(t)} \|f(s) - h \circ \Phi_f(t)\| = \sup_{t\in T} \mathrm{rad}_X(\Phi_f(t)).$$

Thus g is a best approximation of f in V.

COROLLARY 10. *Let X be a dual l.u.c. Banach space. Then every SW-subspace of $C(S, X)$ for which the corresponding φ is open is proximinal.*

We do not know whether in Theorem 9 and Corollary 10 the assumption that φ is open may be dropped. Nor do we know whether the condition that cent_X has a continuous selection is necessary for the proximinality of SW-subspaces. The following theorem gives a necessary condition for the proximinality of such subspaces.

THEOREM 11. *Let F be a compact set in a Banach space X for which $\text{cent}_X(F) = \varnothing$. Then there is a compact Hausdorff space S and an SW-subspace of $C(S, X)$ which is not proximinal.*

Proof. Put $S = F$. Let $T = \{t\}$ be an arbitrary one point set. Put $\varphi(s) = t$ for every $s \in S$. Let $f \in C(S, X)$ be the identity map. We obviously have $\text{dist}(f, V) = \text{rad}_X(F)$. Let $g = h \circ \varphi \in V$ for any $h \in C(T, X)$. Since $\text{cent}_X = \varnothing$ we have $\sup_{s \in S} \|f(s) - g(s)\| = \sup_{s \in S} \|f(s) - h(t)\| > \text{rad}_X(F)$. It follows that g cannot be a best approximation of f.

Garkavi [8] has given an example of a Banach space X and a three-point subset F of X with $\text{cent}_X(F) = \varnothing$. This, together with Theorem 11 provides an example of a space $C(S, X)$ with an SW-subspace which is not proximinal.

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